Malaysian Journal of Mathematical Sciences 19(3): 975–995 (2025)
https://doi.org/10.47836/mjms.19.3.11

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

A Study On Perturbed Newton-type Inequalities For Fractional Operator and Application

Munir, A.*[1], Qayyum, A.*[2,3], Baskonus, H. M.[4], Alsulami, I. K.[5], and Ghareeb, S.[6]

[1]*School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026 People's Republic of China*
[2]*Institute of Mathematical Sciences, Universiti of Malaya, Malaysia*
[3]*Department of Mathematics, University of Southern Punjab Multan Pakistan*
[4]*Department of Mathematics and Science Education, Faculty of Education, Harran University, Sanliurfa, Turkey*
[5]*Department of Science, King Abdulaziz Military Academy (KAMA), Riyadh, Saudi Arabia*
[6]*College of Basic Education - Kuwait*

E-mail: dratherqayyum@um.edu.my
**Corresponding author*

Received: 10 January 2025
Accepted: 24 March 2025

Abstract

Fractional integral inequality is very important in the scientific and engineering disciplines which creates and expands various mathematical methods. In this article, the main aim is to develop the Perturbed Newton-type inequalities by using fractional operator. We derive new identity using the Caputo Fabrizio fractional operator. We use a specific Peano kernel to obtain the Perturbed Newton-type inequalities for the s-φ-convex and φ-quasi-convex functions. Based on this identity, we established several new error bounds and estimations which are appraised by employing some well-known inequalities Hölder and Power-mean. Furthermore, we introduce the refinement of Simpson's second formula, which is not developed by any researcher yet. We also present an application to some special means.

Keywords: perturbed Newton-type inequalities; $s-\varphi$-convex function; fractional integrals; φ-quasi-convex function; power-mean inequality.

1 Introduction

Fractional calculus, also known as the calculus of variations, is indeed a fascinating and important area of mathematics and science. Fractional calculus has gained significant recognition and practical applications in a wide range of scientific disciplines. The theory of inequalities has played a crucial role in the extension and development of mathematics. Mathematical inequalities have permeated numerous areas of science and technology, demonstrating their broad applicability and importance. Mathematical inequalities are also used in the fields of signal process, data science, coding theory, economics, and bioengineering [16]. Indeed, Hermite-Hadamard inequality [13], corrected Euler-Maclaurin type inequalities [19], Newton's inequality [9], Simpson's inequality [6] and Jensen-Mercer inequality [17] are well-known and widely used in various branches of mathematics [26].

Here, we choose the Newton's inequality type inequality for convex functions. We know that there was a deep and intimate connection was discovered between inequalities and the theory of convex functions. Convex functions play a prominent and foundation role in both theoretical and applied studies across various fields of mathematics and its applications. Convexity, with its mathematical elegance and practical utility, permeates numerous aspects of our daily lives, contributing to advancements in medicine, art, and various industries. Due to their importance in inequality theory, convex functions are studied in many academic fields. The well-known definition of a convex function is as follows:

$$\Omega\left(\Delta\Lambda+(1-\Delta)\Psi\right)\leq\Delta\Omega\left(\Lambda\right)+(1-\Delta)\Omega\left(\Psi\right),$$

where $\Omega:I\to\mathbb{R}$ is a convex on I for all $\Lambda,\Psi\in I$ and $\Delta\in[0,1]$.

Many types of integral inequalities exist in the literature. A well-known, one of these is the Simpson type inequality. Simpson's integral inequality is an important result in numerical analysis. In numerical analysis, error bounds and inequalities associated with various numerical integration methods, including Simpson's Rule, are crucial. In all of this years, Thomas Simpson established fundamental methods for numerical integration and estimate of definite integrals that are now known as Simpson's law (1710–1761). But Kepler utilized an identical approximation over a century before, which is because it is often referred to as Kepler's law. Estimates based just on a three-step quadratic kernel are often referred to as Newton-type results as Simpson's method utilizes the three-point Newton-Cotes quadrature rule. The following inequality [6] (which is known as Simpson's inequality) has been studied for (s,m)-convex functions [7], for twice differentiable functions [11] and for (α,m)-convex functions via quantum calculus [29]).

Theorem 1.1. *Let $\Omega:[\Lambda,\Psi]\to\mathbb{R}$ be a four times continuously differentiable mapping on (Λ,Ψ) and $\left\|\Omega^{(4)}\right\|_{\infty}=\sup_{x\in(\Lambda,\Psi)}\left|\Omega^{(4)}\right|<\infty$, then the following inequality holds,*

$$\left|\left[\frac{\Omega\left(\Lambda\right)+\Omega\left(\Psi\right)}{6}+\frac{2}{3}\Omega\left(\frac{\Lambda+\Psi}{2}\right)\right]-\frac{1}{\Psi-\Lambda}\int_{\Lambda}^{\Psi}\Omega\left(x\right)dx\right|\leq\frac{1}{2880}\left\|\Omega^{(4)}\right\|_{\infty}\left(\Psi-\Lambda\right)^{4}.$$

Simpson's type inequality is one of many that are connected with specific quadrature laws is given as follows:

- Simpson quadrature formula (Simpson's 1/3):

$$\left|\frac{\Omega\left(\Lambda\right)}{6}+\frac{4}{6}\Omega\left(\frac{\Lambda+\Psi}{2}\right)+\frac{\Omega\left(\Psi\right)}{6}-\frac{1}{\Psi-\Lambda}\int_{\Lambda}^{\Psi}\Omega\left(u\right)du\right|\leq\frac{\left(\Psi-\Lambda\right)^{4}}{2880}\left\|\Omega^{(4)}\right\|_{\infty}. \tag{1}$$

- Simpson second formula or Newton-Cotes quadrature formula (Simpson's 3/8):

$$\left|\frac{\Omega(\Lambda)}{8}+\frac{3}{8}\Omega\left(\frac{2\Lambda+\Psi}{2}\right)+\frac{3}{8}\Omega\left(\frac{\Lambda+2\Psi}{2}\right)+\frac{\Omega(\Psi)}{8}-\frac{1}{\Psi-\Lambda}\int_{\Lambda}^{\Psi}\Omega(u)\,du\right|\leq\frac{(\Psi-\Lambda)^{4}}{6480}\left\|\Omega^{(4)}\right\|_{\infty}. \tag{2}$$

Fractional analysis has became a popular and influential topic in mathematics due to its ability to address the limitations of traditional calculus and provide a more accurate and versatile framework for modeling and understanding complex phenomena. Numerous authors have used different fractional operators and generalizations of convexity to obtain new error bounds. The development and exploration of new mathematical operators, particularly in the context of fractional calculus, indeed reflect a continuous quest for more effective tools to address real-world problems. In [18], authors used fractional operator for s-convex Function and while in [23], they used a specific operator (k-fractional operator) for (h,m)-convex functions.

The use of fractional operators has indeed been instrumental in conducting numerous studies to explain various physical phenomena and demonstrate wide usage area in mathematical model in biological science, and inequality theory. In fractional calculus, several different types of operators are commonly used such as: the Caputo-Fabrizio fractional derivative [20] and the Riemann-Liouville and Caputo derivatives [27]. Sanli et al. [25] establish the Simpson type inequality using the conformable fractional operator. Butt et al. introduces the refinements of Chebyshev–Pólya–Szegö-type inequalities with the help of generalized fractional integral operators in [1]. Meanwhile, Soontharanon et al. [29] presents the Simpson and Newton's type inequalities via $(\alpha,\,m)$ convex function for quantum calculus. Chen et al. [5] produces new error bounds for Simpson type inequalities for s-convex function. Dragomir et al. [6] obtain the Simpson-type inequalities for differentiable mapping of bounded variation and give some application to special means.

Other than that, Park et al. [22] establish the Simpson-type inequality for differentiable functions whose absolute values are pre-invex. Hezenci et al. [31] prove the Simpson-type inequalities for twice differentiable functions whose absolute values are convex. The Simpson-type inequality is also called Newton-type or (Simpson second formula) if it consists of four points. In Meftah et al. [14], they prove the Newton-type inequalities for differentiable functions whose absolute values are s-convex. Chasreechai et al. [4] derive the Newton-type inequalities using the multiplicative functions. Besides, Butt et al. [2] developed novel Newton type inequalities for majorization. In Saleh et al. [24], they give Newton type inequalities for fractional bi-parameterizes. In addition, Iftikhar et al. [12] provided details about Newton-type inequalities and Siricharuanun et al. [28] provided materials about new Simpson's and Newton's formulas type inequalities.

2 Preliminaries

Definition 2.1. [30] *Suppose $0<s\leq 1$. A function $\Omega: I\subset\mathbb{R}$ is said to be s-φ-convex if the inequality holds:*

$$\Omega(\Delta\Lambda+(1-\Delta)\Psi)\leq\Omega(\Psi)+\Delta^{s}\varphi(\Omega(\Lambda),\Omega(\Psi)),$$

where $\Lambda,\Psi\in I$ and $\Delta\in[0,1]$.

Definition 2.2. [10] *Let $\Omega: I\subset\mathbb{R}$ is said to be φ-quasi-convex if the inequality holds:*

$$\Omega(\Delta\Lambda+(1-\Delta)\Psi)\leq\max[\Omega(\Psi),\Omega(\Psi)+\varphi(\Omega(\Lambda),\Omega(\Psi))],$$

where $\Lambda, \Psi \in I$ *and* $\Delta \in [0,1]$.

Remark 2.1. *If we put* $\varphi(\Lambda, \Psi) = \Lambda - \Psi$ *and* $s = 1$ *in above definition are reduced to the definition of convex and quasi-convex, respectively.*

Iftikhar et al. in [21] establish the Simpson's 3/8 for p-harmonic convex function via a differentiable function as follows:

Theorem 2.1. *Let* $\Omega : [\Lambda, \Psi] \to \mathbb{R}$ *be a differentiable mapping on* I^o *where* $\Lambda, \Psi \in I^o$ *with* $\Lambda < \Psi$. *If* $\Omega' \in L^1[\Lambda, \Psi]$, *then the following inequality holds:*

$$\begin{aligned} &\left| \frac{\Omega(\Lambda)}{8} + \frac{3}{8}\Omega\left(\frac{2\Lambda+\Psi}{2}\right) + \frac{3}{8}\Omega\left(\frac{\Lambda+2\Psi}{2}\right) + \frac{\Omega(\Psi)}{8} - \frac{1}{\Psi-\Lambda}\int_{\Lambda}^{\Psi} \Omega(u)\,du \right| \\ \leq\ & (\Psi-\Lambda)\left[\frac{17}{756}\left(\frac{973\left|\Omega'(\Lambda)\right|^q + 251\left|\Omega'(\Psi)\right|^q}{1224}\right)^{\frac{1}{q}} + \frac{1}{36}\left(\frac{\left|\Omega'(\Lambda)\right|^q + \left|\Omega'(\Psi)\right|^q}{2}\right)^{\frac{1}{q}} \right. \\ &\left. + \frac{17}{756}\left(\frac{251\left|\Omega'(\Lambda)\right|^q + 973\left|\Omega'(\Psi)\right|^q}{1224}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Erden et al. [8] obtained the error bounds of Simpson's second formula for differentiable function.

Theorem 2.2. *Let* $\Omega : [\Lambda, \Psi] \to \mathbb{R}$ *be a differentiable mapping whose derivative is continuously on* (Λ, Ψ) . *Then, for all* $x \in [\Lambda, \Psi]$, *the following inequality holds:*

$$\left| \frac{\Omega(\Lambda)}{8} + \frac{3}{8}\Omega\left(\frac{2\Lambda+\Psi}{2}\right) + \frac{3}{8}\Omega\left(\frac{\Lambda+2\Psi}{2}\right) + \frac{\Omega(\Psi)}{8} - \frac{1}{\Psi-\Lambda}\int_{\Lambda}^{\Psi} \Omega(u)\,du \right| \leq \frac{25(\Psi-\Lambda)}{288}\left\|\Omega\right\|_{\infty}.$$

Let's give some basic definitions in the literature without giving the definitions of fractional integral.

Definition 2.3. *The Euler Gamma function and Euler Beta function are defined:*

$$\Gamma(x) := \int_0^{\infty} \Delta^{x-1} e^{-\Delta} d\Delta,$$

$$\Psi(x,y) := \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1}\, d\Delta.$$

Here, $0 < x, y < \infty$.

We shall need the following integral representations [15]:

$${}_2\Omega_1(\alpha, \beta; \gamma; -z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)} \int_0^1 \frac{\Delta^{\alpha-1}(1-\Delta)^{\gamma-\alpha-1}}{(1+z)^{\beta}} d\Delta,$$

where $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha)$, $|\arg(1+z)| < \Omega$.

Definition 2.4. [3] *Let $H'(\Lambda,\Psi)$ be the Sobolev space of order one defined as,*

$$H^1(\Lambda,\Psi)=\left\{\Omega\in L^2(\Lambda,\Psi):\Omega'\in L^2(\Lambda,\Psi)\right\},$$

where

$$L^2(\Lambda,\Psi)=\left\{\Omega(z):\left(\int_\Lambda^\Psi \Omega^2(z)\,dz\right)^{\frac{1}{2}}<\infty\right\}.$$

Let $\Omega\in H'(\Lambda,\Psi)$, $\Lambda<\Psi$ and $\alpha\in[0,1]$, the nth notion of left derivative in the sense of Caputo-Fabrizio is defined as:

$$\left({}^{CFD}_{\Lambda}D^{\alpha}\Omega\right)(x)=\frac{\beta(\alpha)}{1-\alpha}\int_\Lambda^x \Omega'(\Delta)\,e^{\frac{-\alpha(x-\Delta)^\alpha}{1-\alpha}}d\Delta,$$

$x>\alpha$ and the associated integral operator is,

$$\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(x)=\frac{1-\alpha}{\beta(\alpha)}\Omega(x)+\frac{\alpha}{\beta(\alpha)}\int_\Lambda^x \Omega(\Delta)\,d\Delta,$$

where $\beta(\alpha)>0$ is the normalization function satisfying $\beta(0)=\beta(1)=1$. For $\alpha=0,\alpha=1$, the left derivative is defined as follows, respectively,

$$\left({}^{CFD}_{\Lambda}D^{0}\Omega\right)(x)=\Omega'(x),$$
$$\left({}^{CFD}_{\Lambda}D^{1}\Omega\right)(x)=\Omega(x)-\Omega(\Lambda).$$

For the right derivative operator,

$$\left({}^{CFD}_{\Psi}D^{\alpha}\Omega\right)(x)=\frac{\beta(\alpha)}{1-\alpha}\int_x^\Psi \Omega'(\Delta)\,e^{\frac{-\alpha(\Delta-x)^\alpha}{1-\alpha}}d\Delta,$$

$x<\Psi$ and the associated integral operator is,

$$\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(x)=\frac{1-\alpha}{\beta(\alpha)}\Omega(x)+\frac{\alpha}{\beta(\alpha)}\int_x^\Psi \Omega(\Delta)\,d\Delta.$$

Motivated by the ongoing research, the main goal of this article is to establish a new identity for the Caputo-Fabrizio fractional integral operator with absolute value of the second derivative having s-φ-convex function. By using the novel identity, we obtain the several new bounds of perturbed Newton-type inequalities and to develop many new fractional perturbed Newton-type inequalities for the s-φ-convex, and φ-quasi-convex functions. We claim that, we introduce the refinement of Simpson second formula which was not developed yet. We also include the application to some special means.

3 Perturbed Newton-type Inequalities for Twice Differentiable Function

New perturbed Newton-type Lemma for a Caputo-Fabrizio fractional operator is introduced in this section.

Lemma 3.1. *Let $\Omega : I^o \subset\to \mathbb{R}$ be a twice differentiable mapping on I^o where $\Lambda, \Psi \in I^o$ with $\Lambda < \Psi$. If $\Omega'' \in L^1[\Lambda, \Psi]$, then the following fractional equality holds:*

$$\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)+\Omega(\Psi)\right]$$
$$-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)$$
$$=(\Psi-\Lambda)^2\int_0^1 M(\Delta)\,\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta,$$

where

$$M(\Delta)=\begin{cases}\dfrac{\Delta}{2}\left(\dfrac{1}{4}-\Delta\right), & \Delta\in\left[0,\dfrac{1}{3}\right),\\[2mm] \dfrac{\Delta}{2}(1-\Delta), & \Delta\in\left[\dfrac{1}{3},\dfrac{2}{3}\right),\\[2mm] (1-\Delta)\left(\dfrac{\Delta}{2}-\dfrac{3}{8}\right), & \Delta\in\left[\dfrac{1}{2},1\right].\end{cases}$$

Proof. Let,

$$(\Psi-\Lambda)^2\int_0^1 M(\Delta)\,\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta$$
$$=(\Psi-\Lambda)^2\left[\int_0^{\frac{1}{3}}\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta+\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{\Delta}{2}(1-\Delta)\,\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta\right.$$
$$\left.\int_{\frac{2}{3}}^1(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta\right]$$
$$=I_1+I_2+I_3.$$

Integration by parts, we have,

$$I_1=\int_0^{\frac{1}{3}}\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta$$
$$=\frac{\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)}{\Psi-\Lambda}\Omega'((1-\Delta)\Lambda+\Delta\Psi)\Big|_0^{\frac{1}{3}}-\frac{1}{\Psi-\Lambda}\int_0^{\frac{1}{3}}\left(\frac{1}{8}-\Delta\right)\Omega'((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta$$
$$=\frac{-1}{72(\Psi-\Lambda)}\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)-\frac{1}{\Psi-\Lambda}\int_0^{\frac{1}{3}}\left(\frac{1}{8}-\Delta\right)\Omega'((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta$$
$$=\frac{-1}{72(\Psi-\Lambda)}\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)-\frac{1}{\Psi-\Lambda}\left[\frac{\left(\frac{1}{8}-\Delta\right)}{\Psi-\Lambda}\Omega((1-\Delta)\Lambda+\Delta\Psi)\Big|_0^{\frac{1}{3}}\right.$$
$$\left.+\frac{1}{\Psi-\Lambda}\int_0^{\frac{1}{3}}\Omega((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta\right]$$
$$=\frac{-1}{72(\Psi-\Lambda)}\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)+\frac{5}{24(\Psi-\Lambda)^2}\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+\frac{1}{8(\Psi-\Lambda)^2}\Omega(\Lambda)$$
$$-\frac{1}{(\Psi-\Lambda)^2}\int_0^{\frac{1}{3}}\Omega((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta$$

$$= \frac{-1}{72(\Psi-\Lambda)}\Omega'\left(\frac{2\Lambda+\Psi}{3}\right) + \frac{5}{24(\Psi-\Lambda)^2}\Omega\left(\frac{2\Lambda+\Psi}{3}\right) + \frac{1}{8(\Psi-\Lambda)^2}\Omega(\Lambda)$$
$$- \frac{1}{(\Psi-\Lambda)^3}\int_{\Lambda}^{\frac{2\Lambda+\Psi}{3}} \Omega(u)\,du, \tag{3}$$

and

$$I_2 = \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{\Delta}{2}(1-\Delta)\,\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta$$
$$= \frac{1}{9(\Psi-\Lambda)}\Omega'\left(\frac{\Lambda+2\Psi}{3}\right) - \frac{1}{9(\Psi-\Lambda)}\Omega'\left(\frac{2\Lambda+\Psi}{3}\right) + \frac{1}{6(\Psi-\Lambda)^2}\Omega\left(\frac{\Lambda+2\Psi}{3}\right)$$
$$+ \frac{1}{6(\Psi-\Lambda)^2}\Omega\left(\frac{2\Lambda+\Psi}{3}\right) - \frac{1}{(\Psi-\Lambda)^3}\int_{\frac{2\Lambda+\Psi}{3}}^{\frac{\Lambda+2\Psi}{3}} \Omega(u)\,du. \tag{4}$$

Similarly, we get,

$$I_3 = \int_{\frac{2}{3}}^{1} (1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta$$
$$= \frac{1}{72(\Psi-\Lambda)}\Omega'\left(\frac{\Lambda+2\Psi}{3}\right) + \frac{5}{24(\Psi-\Lambda)^2}\Omega\left(\frac{\Lambda+2\Psi}{3}\right)$$
$$+ \frac{1}{8(\Psi-\Lambda)^2}\Omega(\Psi) - \frac{1}{(\Psi-\Lambda)^3}\int_{\frac{\Lambda+2\Psi}{3}}^{\Psi} \Omega(u)\,du. \tag{5}$$

Adding the equalities (3), (4) and (5), we get,

$$I_1 + I_2 + I_3$$
$$= \frac{-1}{72(\Psi-\Lambda)}\Omega'\left(\frac{2\Lambda+\Psi}{3}\right) + \frac{5}{24(\Psi-\Lambda)^2}\Omega\left(\frac{2\Lambda+\Psi}{3}\right) + \frac{1}{8(\Psi-\Lambda)^2}\Omega(\Lambda)$$
$$- \frac{1}{(\Psi-\Lambda)^3}\int_{\Lambda}^{\frac{2\Lambda+\Psi}{3}} \Omega(u)\,du + \frac{1}{9(\Psi-\Lambda)}\Omega'\left(\frac{\Lambda+2\Psi}{3}\right) - \frac{1}{9(\Psi-\Lambda)}\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)$$
$$+ \frac{1}{6(\Psi-\Lambda)^2}\Omega\left(\frac{\Lambda+2\Psi}{3}\right) + \frac{1}{6(\Psi-\Lambda)^2}\Omega\left(\frac{2\Lambda+\Psi}{3}\right) - \frac{1}{(\Psi-\Lambda)^3}\int_{\frac{2\Lambda+\Psi}{3}}^{\frac{\Lambda+2\Psi}{3}} \Omega(u)\,du$$
$$+ \frac{1}{72(\Psi-\Lambda)}\Omega'\left(\frac{\Lambda+2\Psi}{3}\right) + \frac{5}{24(\Psi-\Lambda)^2}\Omega\left(\frac{\Lambda+2\Psi}{3}\right) + \frac{1}{8(\Psi-\Lambda)^2}\Omega(\Psi)$$
$$- \frac{1}{(\Psi-\Lambda)^3}\int_{\frac{\Lambda+2\Psi}{3}}^{\Psi} \Omega(u)\,du$$
$$= \frac{-1}{8(\Psi-\Lambda)}\Omega'\left(\frac{2\Lambda+\Psi}{3}\right) + \frac{1}{8(\Psi-\Lambda)}\Omega'\left(\frac{\Lambda+2\Psi}{3}\right) + \frac{1}{8(\Psi-\Lambda)^2}\Omega(\Lambda)$$
$$+ \frac{1}{8(\Psi-\Lambda)^2}\Omega(\Psi) + \frac{3}{8(\Psi-\Lambda)^2}\Omega\left(\frac{2\Lambda+\Psi}{3}\right) + \frac{3}{8(\Psi-\Lambda)^2}\Omega\left(\frac{\Lambda+2\Psi}{3}\right)$$
$$- \frac{1}{(\Psi-\Lambda)^3}\int_{\Lambda}^{\Psi} \Omega(u)\,du. \tag{6}$$

Multiplying by $(\Psi-\Lambda)^2$ with equality (6) and subtracting $\frac{2(1-\alpha)}{\beta(\alpha)(\Psi-\Lambda)}\Omega(k)$, we have,

$$
\begin{aligned}
&(I_1+I_2+I_3)(\Psi-\Lambda)^2-\frac{2(1-\alpha)}{\beta(\alpha)(\Psi-\Lambda)}\Omega(k)\\
&=\frac{-(\Psi-\Lambda)}{8}\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)+\frac{(\Psi-\Lambda)}{8}\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)+\frac{1}{8}(\Omega(\Lambda)+\Omega(\Psi))+\frac{3}{8}\Omega\left(\frac{2\Lambda+\Psi}{3}\right)\\
&\quad+\frac{3}{8}\Omega\left(\frac{\Lambda+2\Psi}{3}\right)-\frac{1}{(\Psi-\Lambda)}\int_\Lambda^\Psi\Omega(u)\,du-\frac{2(1-\alpha)}{\beta(\alpha)(\Psi-\Lambda)}\Omega(k)\\
&=\frac{-(\Psi-\Lambda)}{8}\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)+\frac{(\Psi-\Lambda)}{8}\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)+\frac{1}{8}(\Omega(\Lambda)+\Omega(\Psi))\\
&\quad+\frac{3}{8}\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+\frac{3}{8}\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\\
&\quad-\frac{1}{(\Psi-\Lambda)}\left(\frac{\alpha}{\beta(\alpha)}\int_\Lambda^k\Omega(u)\,du-\frac{(1-\alpha)}{\beta(\alpha)}\Omega(k)+\frac{\alpha}{\beta(\alpha)}\int_k^\Psi\Omega(u)\,du-\frac{(1-\alpha)}{\beta(\alpha)}\Omega(k)\right)\\
&=\frac{-(\Psi-\Lambda)}{8}\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)+\frac{(\Psi-\Lambda)}{8}\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)+\frac{1}{8}(\Omega(\Lambda)+\Omega(\Psi))\\
&\quad+\frac{3}{8}\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+\frac{3}{8}\Omega\left(\frac{\Lambda+2\Psi}{3}\right)-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right].
\end{aligned}
$$

Thus, we have,

$$
\begin{aligned}
&\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\\
&\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\\
&\quad=(\Psi-\Lambda)^2\int_0^1 M(\Delta)\,\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta.
\end{aligned}
$$

The proof of Lemma 3.1 is completed. □

4 Perturbed Newton-type Inequalities for s-φ-convex

In this section, we present the fractional Perturbed Newton-type inequalities for s-φ-convex function.

Theorem 4.1. *Under the assumption of Lemma 3.1. If $|\Omega''|$ is s-φ-convex on $[\Lambda,\Psi]$, then the following fractional inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.\\
&\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|\\
&\le(\Psi-\Lambda)^2\left[\left(\frac{19}{10368}\right)|\Omega''(\Lambda)|+\left(\frac{2^{2s+3}s+3^{s+3}-2^{2s+3}}{2^{2s+6}3^{s+3}(5s+s^s+6)}\right)\varphi(|\Omega''(\Lambda)|,|\Omega''(\Psi)|)\right.\\
&\left.+\left(\frac{13}{324}\right)|\Omega''(\Lambda)|+\left(\frac{5\times2^{s+2}-2s+2^{s+2}s-7}{2\times3^{s+3}(s+2)(s+3)}\right)\varphi(|\Omega''(\Lambda)|,|\Omega''(\Psi)|)\right.
\end{aligned}
$$

$$+\left(\frac{19}{10368}\right)|\Omega''(\Lambda)|-\frac{1}{(s+1)(s+2)(s+3)}\left(3^{2s+5}s-15\times 2^{3s+4}s-2^{3s+4}s^2+3^{2s+7}\right.$$
$$\left.+2^{2s+3}3^{s+3}s+\frac{1}{2^{2s+6}3^{s+3}}\left(5\times 2^{2s+3}s+3^{s+3}+17\times 2^{3s+6}\right)\right)\varphi\left(|\Omega''(\Lambda)|,|\Omega''(\Psi)|\right)\Bigg].$$

Proof. By using the Lemma 3.1, since s-φ-convexity of $|\Omega''|$, we have,

$$\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.\\
&\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|\\
&\leq(\Psi-\Lambda)^2\left|\int_0^1 M(\Delta)\,\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta\right|\\
&\leq(\Psi-\Lambda)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right||\Omega''((1-\Delta)\Lambda+\Delta\Psi)|\,d\Delta\right.\\
&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right||\Omega''((1-\Delta)\Lambda+\Delta\Psi)|\,d\Delta\\
&\quad\left.+\int_{\frac{2}{3}}^1\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right||\Omega''((1-\Delta)\Lambda+\Delta\Psi)|\,d\Delta\right]\\
&\leq(\Psi-\Lambda)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|\left(|\Omega''(\Lambda)|+\Delta^s\varphi\left(|\Omega''(\Lambda)|,|\Omega''(\Psi)|\right)\right)d\Delta\right.\\
&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|\left(|\Omega''(\Lambda)|+\Delta^s\varphi\left(|\Omega''(\Lambda)|,|\Omega''(\Psi)|\right)\right)d\Delta\\
&\quad\left.+\int_{\frac{2}{3}}^1\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|\left(|\Omega''(\Lambda)|+\Delta^s\varphi\left(|\Omega''(\Lambda)|,|\Omega''(\Psi)|\right)\right)d\Delta\right]\\
&=(\Psi-\Lambda)^2(D_1+D_2+D_3),
\end{aligned}\tag{7}$$

where

$$\begin{aligned}
D_1&=\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|\left(|\Omega''(\Lambda)|+\Delta^s\varphi\left(|\Omega''(\Lambda)|,|\Omega''(\Psi)|\right)\right)d\Delta\\
&=\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right||\Omega''(\Lambda)|\,d\Delta+\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|\Delta^s\varphi\left(|\Omega''(\Lambda)|,|\Omega''(\Psi)|\right)d\Delta\\
&=\left(\frac{19}{10368}\right)|\Omega''(\Lambda)|+\left(\frac{2^{2s+3}s+3^{s+3}-2^{2s+3}}{2^{2s+6}3^{s+3}(5s+s^s+6)}\right)\varphi\left(|\Omega''(\Lambda)|,|\Omega''(\Psi)|\right).
\end{aligned}\tag{8}$$

$$\begin{aligned}
D_2&=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|\left(|\Omega''(\Lambda)|+\Delta^s\varphi\left(|\Omega''(\Lambda)|,|\Omega''(\Psi)|\right)\right)d\Delta\\
&=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right||\Omega''(\Lambda)|\,d\Delta+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|\Delta^s\varphi\left(|\Omega''(\Lambda)|,|\Omega''(\Psi)|\right)d\Delta\\
&=\left(\frac{13}{324}\right)|\Omega''(\Lambda)|+\left(\frac{5\times 2^{s+2}-2s+2^{s+2}s-7}{2\times 3^{s+3}(s+2)(s+3)}\right)\varphi\left(|\Omega''(\Lambda)|,|\Omega''(\Psi)|\right).
\end{aligned}\tag{9}$$

$$D_3=\int_{\frac{2}{3}}^1\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|\left(|\Omega''(\Lambda)|+\Delta^s\varphi\left(|\Omega''(\Lambda)|,|\Omega''(\Psi)|\right)\right)d\Delta$$

$$
\begin{aligned}
&= \int_{\frac{2}{3}}^{1} \left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right| \left|\Omega''(\Lambda)\right| d\Delta + \int_{\frac{2}{3}}^{1} \left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right| \Delta^{s} \varphi\left(\left|\Omega''(\Lambda)\right|, \left|\Omega''(\Psi)\right|\right) d\Delta \\
&= \left(\frac{19}{10368}\right)\left|\Omega''(\Lambda)\right| - \frac{1}{(s+1)(s+2)(s+3)} \\
&\quad \times \left(3^{2s+5}s - 15 \times 2^{3s+4}s - 2^{3s+4}s^2 + 3^{2s+7} + 2^{2s+3}3^{s+3}s \right. \\
&\quad \left. + \frac{1}{2^{2s+6}3^{s+3}}\left(5 \times 2^{2s+3}s + 3^{s+3} + 17 \times 2^{3s+6}\right)\right) \varphi\left(\left|\Omega''(\Lambda)\right|, \left|\Omega''(\Psi)\right|\right). \qquad (10)
\end{aligned}
$$

Using the equalities (8), (9) and (10) in (7), we get the required result.

This completes the proof. □

Corollary 4.1. *If we choose $s = 1$ in Theorem 4.1, then we get,*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right. \\
&\left.\left.+\Omega(\Psi)\right] - \frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}_{\Lambda}^{CF}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I_{\Psi}^{\alpha}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right| \\
&\leq (\Psi-\Lambda)^2\left[\frac{125\left|\Omega''(\Lambda)\right|+27\varphi\left(\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Psi)\right|\right)}{82944}+\frac{26\left|\Omega''(\Lambda)\right|+13\left|\Omega''(\Psi)\right|}{648}\right. \\
&\left.+\frac{152\left|\Omega''(\Lambda)\right|+125\varphi\left(\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Psi)\right|\right)}{82944}\right].
\end{aligned}
$$

Corollary 4.2. *Applying again convexity on Corollary 4.1, we get,*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right. \\
&\left.\left.+\Omega(\Psi)\right] - \frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}_{\Lambda}^{CF}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I_{\Psi}^{\alpha}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right| \\
&\leq (\Psi-\Lambda)^2\left(\frac{227}{5184}\right)\left[\left|\Omega''(\Lambda)\right|+\frac{1}{2}\varphi\left(\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Psi)\right|\right)\right].
\end{aligned}
$$

Corollary 4.3. *If we choose $\varphi\left(\left|\Omega''(\Lambda)\right|, \left|\Omega''(\Psi)\right|\right) = \left|\Omega''(\Psi)\right| - \left|\Omega''(\Lambda)\right|$ in Corollary 4.2, then we get,*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right. \\
&\left.\left.+\Omega(\Psi)\right] - \frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}_{\Lambda}^{CF}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I_{\Psi}^{\alpha}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right| \\
&\leq (\Psi-\Lambda)^2\left(\frac{227}{10368}\right)\left(\left|\Omega''(\Lambda)\right|+\left|\Omega''(\Psi)\right|\right).
\end{aligned}
$$

Theorem 4.2. *Under the assumption of Lemma 3.1. If $\left|\Omega''\right|^{q}$ is s-φ-convex on $[\Lambda, \Psi]$ and $q > 1$, then the following fractional inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right. \\
&\left.\left.+\Omega(\Psi)\right] - \frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}_{\Lambda}^{CF}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I_{\Psi}^{\alpha}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|
\end{aligned}
$$

$$\leq (\Psi-\Lambda)^2\left[(C_1)^{\frac{1}{r}}\left(\frac{6}{18}\left|\Omega''(\Lambda)\right|^q+\frac{\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)}{3^{s+1}(s+1)}\right)^{\frac{1}{q}}\right.$$
$$+(C_2)^{\frac{1}{r}}\left(\frac{2}{6}\left|\Omega''(\Lambda)\right|^q+\frac{3^{-s-1}\left(-1+2^{1+s}\right)\left(\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right)}{(s+1)}\right)^{\frac{1}{q}}$$
$$\left.+(C_3)^{\frac{1}{r}}\left(\frac{6}{18}\left|\Omega''(\Lambda)\right|^q+\frac{2^s\times 3^{-s-1}\left(-2+2^{-s}\times 3^{1+s}\right)\left(\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right)}{(s+1)}\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using the Lemma 3.1, with the help of Hölder inequality and s-φ-convexity of $|\Omega''|^q$, we have,

$$\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.$$
$$\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|$$
$$\leq (\Psi-\Lambda)^2\left|\int_0^1 M(\Delta)\,\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta\right|$$
$$\leq (\Psi-\Lambda)^2\left[\left(\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|^r d\Delta\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\left|\Omega''((1-\Delta)\Lambda+\Delta\Psi)\right|^q d\Delta\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|^r d\Delta\right)^{\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\Omega''((1-\Delta)\Lambda+\Delta\Psi)\right|^q d\Delta\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|^r d\Delta\right)^{\frac{1}{r}}\left(\int_{\frac{2}{3}}^{1}\left|\Omega''((1-\Delta)\Lambda+\Delta\Psi)\right|^q d\Delta\right)^{\frac{1}{q}}\right]$$
$$\leq (\Psi-\Lambda)^2\left[\left(\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|^r d\Delta\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}(1-\Delta)\left|\Omega''(\Lambda)\right|^q d\Delta\right.\right.$$
$$\left.+\int_0^{\frac{1}{3}}\Delta^s\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)d\Delta\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|^r d\Delta\right)^{\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}(1-\Delta)\left|\Omega''(\Lambda)\right|^q d\Delta\right.$$
$$\left.+\int_{\frac{1}{3}}^{\frac{2}{3}}\Delta^s\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)d\Delta\right)^{\frac{1}{q}}+\left(\int_{\frac{2}{3}}^{1}\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|^r d\Delta\right)^{\frac{1}{r}}$$
$$\left.\left(\int_{\frac{2}{3}}^{1}(1-\Delta)\left|\Omega''(\Lambda)\right|^q d\Delta+\int_{\frac{2}{3}}^{1}\Delta^s\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)d\Delta\right)^{\frac{1}{q}}\right].$$

Employing the s-φ-convexity of $|\Omega''|^q$, we have,

$$\left(\int_0^{\frac{1}{3}}(1-\Delta)\left|\Omega''(\Lambda)\right|^q d\Delta+\int_0^{\frac{1}{3}}\Delta^s\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)d\Delta\right)^{\frac{1}{q}}$$

$$= \left(\frac{6}{18} \left| \Omega'' (\Lambda) \right|^q + \frac{\varphi \left(\left| \Omega'' (\Lambda) \right|^q, \left| \Omega'' (\Psi) \right|^q \right)}{3^{s+1} (s+1)} \right)^{\frac{1}{q}}.$$

Also, we get,

$$\left(\int_{\frac{1}{3}}^{\frac{2}{3}} (1-\Delta) \left| \Omega'' (\Lambda) \right|^q d\Delta + \int_{\frac{1}{3}}^{\frac{2}{3}} \Delta^s \varphi \left(\left| \Omega'' (\Lambda) \right|^q, \left| \Omega'' (\Psi) \right|^q \right) d\Delta \right)^{\frac{1}{q}}$$
$$= \left(\frac{2}{6} \left| \Omega'' (\Lambda) \right|^q + \frac{3^{-s-1} \left(-1 + 2^{1+s} \right) \left(\varphi \left(\left| \Omega'' (\Lambda) \right|^q, \left| \Omega'' (\Psi) \right|^q \right) \right)}{(s+1)} \right)^{\frac{1}{q}},$$

and

$$\left(\int_{\frac{2}{3}}^{1} (1-\Delta) \left| \Omega'' (\Lambda) \right|^q d\Delta + \int_{\frac{2}{3}}^{1} \Delta^s \varphi \left(\left| \Omega'' (\Lambda) \right|^q, \left| \Omega'' (\Psi) \right|^q \right) d\Delta \right)^{\frac{1}{q}}$$
$$= \left(\frac{6}{18} \left| \Omega'' (\Lambda) \right|^q + \frac{2^s \times 3^{-s-1} \left(-2 + 2^{-s} \times 3^{1+s} \right) \left(\varphi \left(\left| \Omega'' (\Lambda) \right|^q, \left| \Omega'' (\Psi) \right|^q \right) \right)}{(s+1)} \right)^{\frac{1}{q}}.$$

We also have,

$$C_1 = \int_0^{\frac{1}{3}} \left| \frac{\Delta}{2} \left(\frac{1}{4} - \Delta \right) \right|^r d\Delta \tag{11}$$
$$= \int_0^{\frac{1}{4}} \left(\frac{\Delta}{2} \left(\frac{1}{4} - \Delta \right) \right)^r d\Delta + \int_{\frac{1}{4}}^{\frac{1}{3}} \left(\frac{\Delta}{2} \left(\Delta - \frac{1}{4} \right) \right)^r d\Delta$$
$$= \frac{1}{8^r} \left[\int_0^{\frac{1}{4}} \left(\Delta (1-4\Delta) \right)^r d\Delta + \int_0^{\frac{1}{3}} \left(\Delta (4\Delta - 1) \right)^r d\Delta - \int_0^{\frac{1}{4}} \left(\Delta (4\Delta - 1) \right)^r d\Delta \right]$$
$$= \frac{1}{8^r} \left[\frac{B(r+1, r+1)}{4^{r+1}} + \frac{-1^r}{3^{r+1} (r+1)} \times_2 \Omega_1 \left(-r, r+1; r+2; \frac{4}{3} \right) \right.$$
$$\left. + \frac{(-1^r) \times B(r+1, r+1)}{4^{r+1}} \times_2 \Omega_1 (-r, r+1; r+2; 1) \right].$$
$$C_2 = \int_{\frac{1}{3}}^{\frac{2}{3}} \left| \frac{\Delta}{2} (1-\Delta) \right|^r d\Delta$$
$$= \frac{1}{2^r} \left[\int_0^{\frac{2}{3}} \Delta^r (1-\Delta)^r d\Delta - \int_0^{\frac{1}{3}} \Delta^r (1-\Delta)^r d\Delta \right]$$
$$= \frac{1}{3^{r+1} (r+1)} \left[2 \times_2 \Omega_1 \left(-r, r+1; r+2; \frac{4}{3} \right) \right].$$
$$C_3 = \int_{\frac{2}{3}}^{1} \left| (1-\Delta) \left(\frac{\Delta}{2} - \frac{3}{8} \right) \right|^r d\Delta$$
$$= \int_{\frac{2}{3}}^{\frac{3}{4}} \left[(1-\Delta) \left(\frac{3}{8} - \frac{\Delta}{2} \right) \right]^r d\Delta + \int_{\frac{3}{4}}^{1} \left[(1-\Delta) \left(\frac{\Delta}{2} - \frac{3}{8} \right) \right]^r d\Delta$$
$$= \left(\frac{3}{8} \right)^r \left[\frac{1}{36^r \times 12} B(r+1, 1) \times_2 \Omega_1 \left(-r, r+1; r+2; \frac{-1}{3} \right) + \frac{1}{4^{r+1} 3^r} B(r+1, r+1) \right].$$

This completes the proof. □

Corollary 4.4. *If we choose $s=1$ in Theorem 4.2, then we get,*

$$\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.\\
&\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|\\
&\leq(\Psi-\Lambda)^2\left[(C_1)^{\frac{1}{r}}\left(\frac{6\left|\Omega''(\Lambda)\right|^q+\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)}{18}\right)^{\frac{1}{q}}\right.\\
&+(C_2)^{\frac{1}{r}}\left(\frac{2\left|\Omega''(\Lambda)\right|^q+\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)}{6}\right)^{\frac{1}{q}}\\
&\left.+(C_3)^{\frac{1}{r}}\left(\frac{6\left|\Omega''(\Lambda)\right|^q+5\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)}{18}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where C_i for $i=1,2,3$ denoted as in (11).

Corollary 4.5. *If we choose $\varphi\left(\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Psi)\right|\right)=\left|\Omega''(\Psi)\right|-\left|\Omega''(\Lambda)\right|$ in Corollary 4.4, then we get,*

$$\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.\\
&\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|\\
&\leq(\Psi-\Lambda)^2\left[(C_1)^{\frac{1}{r}}\left(\frac{5\left|\Omega''(\Lambda)\right|^q+\left|\Omega''(\Psi)\right|^q}{18}\right)^{\frac{1}{q}}+(C_2)^{\frac{1}{r}}\left(\frac{\left|\Omega''(\Lambda)\right|^q+\left|\Omega''(\Psi)\right|^q}{6}\right)^{\frac{1}{q}}\right.\\
&\left.+(C_3)^{\frac{1}{r}}\left(\frac{\left|\Omega''(\Lambda)\right|^q+5\left|\Omega''(\Psi)\right|^q}{18}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where C_i for $i=1,2,3$ denoted as in (11).

Theorem 4.3. *Under the assumption of Lemma 3.1. If $\left|\Omega''\right|^q$ is s-φ-convex on $[\Lambda,\Psi]$ and $q\geq 1$, then the following fractional inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.\\
&\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|\\
&\leq(\Psi-\Lambda)^2\left[\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{152\left|\Omega''(\Lambda)\right|^q}{82944}+\frac{27\left(2^{2s+3}s+3^{s+3}-2^{2s+3}\right)}{2^{2s+6}3^{s+3}(5s+s^s+6)}\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{13}{324}\right)^{1-\frac{1}{q}}\left(\frac{26\left|\Omega''(\Lambda)\right|^q}{648}+\frac{\left(5\times 2^{s+2}-2s+2^{s+2}s-7\right)}{2\times 3^{s+3}(s+2)(s+3)}\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right)^{\frac{1}{q}}\\
&+\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{152\left|\Omega''(\Lambda)\right|^q}{82944}-\frac{1}{(s+1)(s+2)(s+3)}\left(3^{2s+5}s-15\times 2^{3s+4}s-2^{3s+4}s^2+3^{2s+7}\right.\right.\\
&\left.\left.\left.+2^{2s+3}3^{s+3}s+\frac{1}{2^{2s+6}3^{s+3}}\left(5\times 2^{2s+3}s+3^{s+3}+17\times 2^{3s+6}\right)\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right)\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using the Lemma 3.1, with the help of power-mean inequality$s$-φ-convexity of $|\Omega''|^q$, we have,

$$
\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.\\
&\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|\\
&\leq(\Psi-\Lambda)^2\left|\int_0^1 M(\Delta)\,\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta\right|\\
&\leq(\Psi-\Lambda)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|\left|\Omega''((1-\Delta)\Lambda+\Delta\Psi)\right|d\Delta+\int_{\frac{1}{3}}^{\frac{2}{3}}\right.\\
&\quad\left.+\left|\frac{\Delta}{2}(1-\Delta)\right|\left|\Omega''((1-\Delta)\Lambda+\Delta\Psi)\right|d\Delta\int_{\frac{2}{3}}^1\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|\left|\Omega''((1-\Delta)\Lambda+\Delta\Psi)\right|d\Delta\right]\\
&\leq(\Psi-\Lambda)^2\left[\left(\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|d\Delta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|\left|\Omega''((1-\Delta)\Lambda+\Delta\Psi)\right|^q d\Delta\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|d\Delta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|\left|\Omega''((1-\Delta)\Lambda+\Delta\Psi)\right|^q d\Delta\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{2}{3}}^1\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|d\Delta\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^1\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|\left|\Omega''((1-\Delta)\Lambda+\Delta\Psi)\right|^q d\Delta\right)^{\frac{1}{q}}\right]\\
&\leq(\Psi-\Lambda)^2\left[\left(\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|d\Delta\right)^{1-\frac{1}{q}}\right.\\
&\quad\times\left(\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|\left(|\Omega''(\Lambda)|^q+\Delta^s\varphi\left(|\Omega''(\Lambda)|^q,|\Omega''(\Psi)|^q\right)\right)d\Delta\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|d\Delta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|\left(|\Omega''(\Lambda)|^q+\Delta^s\varphi\left(|\Omega''(\Lambda)|^q,|\Omega''(\Psi)|^q\right)\right)d\Delta\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{2}{3}}^1\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|d\Delta\right)^{1-\frac{1}{q}}\\
&\quad\left.\times\left(\int_{\frac{2}{3}}^1\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|\left(|\Omega''(\Lambda)|^q+\Delta^s\varphi\left(|\Omega''(\Lambda)|^q,|\Omega''(\Psi)|^q\right)\right)d\Delta\right)^{\frac{1}{q}}\right]\\
&\leq(\Psi-\Lambda)^2\left[\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{152\,|\Omega''(\Lambda)|^q}{82944}+\frac{27\left(2^{2s+3}s+3^{s+3}-2^{2s+3}\right)}{2^{2s+6}3^{s+3}(5s+s^s+6)}\varphi\left(|\Omega''(\Lambda)|^q,|\Omega''(\Psi)|^q\right)\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{13}{324}\right)^{1-\frac{1}{q}}\left(\frac{26\,|\Omega''(\Lambda)|^q}{648}+\frac{5\times2^{s+2}-2s+2^{s+2}s-7}{2\times3^{s+3}(s+2)(s+3)}\varphi\left(|\Omega''(\Lambda)|^q,|\Omega''(\Psi)|^q\right)\right)^{\frac{1}{q}}\right.
\end{aligned}
$$

$$+\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{152\left|\Omega''(\Lambda)\right|^q}{82944}-\frac{1}{(s+1)(s+2)(s+3)}\right.$$
$$\times\left(3^{2s+5}s-15\times 2^{3s+4}s-2^{3s+4}s^2+3^{2s+7}+2^{2s+3}3^{s+3}s\right.$$
$$\left.\left.+\frac{1}{2^{2s+6}3^{s+3}}\left(5\times 2^{2s+3}s+3^{s+3}+17\times 2^{3s+6}\right)\right)\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right)^{\frac{1}{q}}\Bigg].$$

This completes the proof. □

Corollary 4.6. *If we choose $s=1$ in Theorem 4.3, then we get,*

$$\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.$$
$$\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}_{\Lambda}^{CF}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I_{\Psi}^{\alpha}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|$$
$$\leq(\Psi-\Lambda)^2\left[\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{152\left|\Omega''(\Lambda)\right|^q+27\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)}{82944}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{13}{324}\right)^{1-\frac{1}{q}}\left(\frac{26\left|\Omega''(\Lambda)\right|^q+13\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)}{648}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{152\left|\Omega''(\Lambda)\right|^q+125\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)}{82944}\right)^{\frac{1}{q}}\right].$$

Corollary 4.7. *If we choose $\varphi\left(\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Psi)\right|\right)=\left|\Omega''(\Psi)\right|-\left|\Omega''(\Lambda)\right|$ in Corollary 4.6, then we get,*

$$\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.$$
$$\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}_{\Lambda}^{CF}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I_{\Psi}^{\alpha}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|$$
$$\leq(\Psi-\Lambda)^2\left[\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{125\left|\Omega''(\Lambda)\right|^q+27\left|\Omega''(\Psi)\right|^q}{82944}\right)^{\frac{1}{q}}+\left(\frac{13}{324}\right)^{1-\frac{1}{q}}\right.$$
$$\left.\left(\frac{13\left|\Omega''(\Lambda)\right|^q+13\left|\Omega''(\Psi)\right|^q}{648}\right)^{\frac{1}{q}}+\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{27\left|\Omega''(\Lambda)\right|^q+125\left|\Omega''(\Psi)\right|^q}{82944}\right)^{\frac{1}{q}}\right].$$

5 Perturbed Newton-type Inequalities for φ-quasi-convex

In this section, we obtain the new errors bounds for perturbed Newton-type inequalities via φ-quasi-convex function.

Theorem 5.1. *Under the assumption of Lemma 3.1. If $|\Omega''|$ is φ-quasi-convex on $[\Lambda,\Psi]$, then the following fractional inequality holds:*

$$\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.$$
$$\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}_{\Lambda}^{CF}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I_{\Psi}^{\alpha}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|$$

$$\leq (\Psi-\Lambda)^2\left(\frac{227}{5184}\right)\max\left[\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Lambda)\right|+\varphi\left(\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Psi)\right|\right)\right].$$

Proof. By using the Lemma 3.1, we have,

$$\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.\\
&\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|\\
&\leq(\Psi-\Lambda)^2\left|\int_0^1 M(\Delta)\,\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta\right|\\
&\leq(\Psi-\Lambda)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|\left|\Omega''((1-\Delta)\Lambda+\Delta\Psi)\right|d\Delta+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|\right.\\
&\left.+\left|\Omega''((1-\Delta)\Lambda+\Delta\Psi)\right|d\Delta\int_{\frac{2}{3}}^1\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|\left|\Omega''((1-\Delta)\Lambda+\Delta\Psi)\right|d\Delta\right]\\
&\leq(\Psi-\Lambda)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|\max\left[\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Lambda)\right|+\varphi\left(\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Psi)\right|\right)\right]d\Delta\right.\\
&+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|\max\left[\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Lambda)\right|+\varphi\left(\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Psi)\right|\right)\right]d\Delta\\
&\left.+\int_{\frac{2}{3}}^1\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|\max\left[\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Lambda)\right|+\varphi\left(\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Psi)\right|\right)\right]d\Delta\right]\\
&\leq(\Psi-\Lambda)^2\left(\frac{227}{5184}\right)\max\left[\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Lambda)\right|+\varphi\left(\left|\Omega''(\Lambda)\right|,\left|\Omega''(\Psi)\right|\right)\right].
\end{aligned}$$

This completes the proof. □

Theorem 5.2. *Under the assumption of Lemma 3.1. If $|\Omega''|^q$ is φ-quasi-convex on $[\Lambda,\Psi]$ and $q\geq 1$, then the following fractional inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.\\
&\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|\\
&\leq(\Psi-\Lambda)^2\left(\frac{227}{5184}\right)\left(\max\left[\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Lambda)\right|^q+\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right]^{\frac{1}{q}}\right).
\end{aligned}$$

Proof. By using the Lemma 3.1, we have,

$$\begin{aligned}
&\left|\frac{1}{8}\left[(\Psi-\Lambda)\left(\Omega'\left(\frac{\Lambda+2\Psi}{3}\right)-\Omega'\left(\frac{2\Lambda+\Psi}{3}\right)\right)+\Omega(\Lambda)+3\Omega\left(\frac{2\Lambda+\Psi}{3}\right)+3\Omega\left(\frac{\Lambda+2\Psi}{3}\right)\right.\right.\\
&\left.\left.+\Omega(\Psi)\right]-\frac{\beta(\alpha)}{\alpha(\Psi-\Lambda)}\left[\left({}^{CF}_{\Lambda}I^{\alpha}\Omega\right)(k)+\left({}^{CF}I^{\alpha}_{\Psi}\Omega\right)(k)\right]+\frac{2(1-\alpha)}{\alpha(\Psi-\Lambda)}\Omega(k)\right|\\
&\leq(\Psi-\Lambda)^2\left|\int_0^1 M(\Delta)\,\Omega''((1-\Delta)\Lambda+\Delta\Psi)\,d\Delta\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq (\Psi-\Lambda)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|\left|\Omega''\left((1-\Delta)\Lambda+\Delta\Psi\right)\right|d\Delta+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|\right.\\
&\quad\left.+\left|\Omega''\left((1-\Delta)\Lambda+\Delta\Psi\right)\right|d\Delta\int_{\frac{2}{3}}^{1}\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|\left|\Omega''\left((1-\Delta)\Lambda+\Delta\Psi\right)\right|d\Delta\right]\\
&\leq (\Psi-\Lambda)^2\left[\left(\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|d\Delta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|\left|\Omega''\left((1-\Delta)\Lambda+\Delta\Psi\right)\right|^q d\Delta\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|d\Delta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|\left|\Omega''\left((1-\Delta)\Lambda+\Delta\Psi\right)\right|^q d\Delta\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{2}{3}}^{1}\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|d\Delta\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|\left|\Omega''\left((1-\Delta)\Lambda+\Delta\Psi\right)\right|^q d\Delta\right)^{\frac{1}{q}}\right]\\
&\leq (\Psi-\Lambda)^2\left[\left(\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|d\Delta\right)^{1-\frac{1}{q}}\right.\\
&\quad\times\left(\int_0^{\frac{1}{3}}\left|\frac{\Delta}{2}\left(\frac{1}{4}-\Delta\right)\right|\max\left[\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Lambda)\right|^q+\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right]d\Delta\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|d\Delta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\Delta}{2}(1-\Delta)\right|\max[\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Lambda)\right|^q\right.\\
&\quad\left.+\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)]d\Delta\right)^{\frac{1}{q}}+\left(\int_{\frac{2}{3}}^{1}\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|d\Delta\right)^{1-\frac{1}{q}}\\
&\quad\left.\times\left(\int_{\frac{2}{3}}^{1}\left|(1-\Delta)\left(\frac{\Delta}{2}-\frac{3}{8}\right)\right|\max\left[\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Lambda)\right|^q+\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right]d\Delta\right)^{\frac{1}{q}}\right]\\
&\leq (\Psi-\Lambda)^2\left[\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\left(\frac{19}{10368}\right)\max\left[\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Lambda)\right|^q+\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right]\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{13}{324}\right)^{1-\frac{1}{q}}\left(\left(\frac{13}{324}\right)\max\left[\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Lambda)\right|^q+\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right]\right)^{\frac{1}{q}}\\
&\quad\left.\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\left(\frac{19}{10368}\right)\max\left[\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Lambda)\right|^q+\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right]\right)^{\frac{1}{q}}\right]\\
&\leq (\Psi-\Lambda)^2\left(\frac{227}{5184}\right)\left(\max\left[\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Lambda)\right|^q+\varphi\left(\left|\Omega''(\Lambda)\right|^q,\left|\Omega''(\Psi)\right|^q\right)\right]^{\frac{1}{q}}\right).
\end{aligned}
$$

This completes the proof. □

6 Application to Some Special Means

(a) The Arithmetic mean:

$$A=A(\Lambda,\Psi):=\frac{\Lambda+\Psi}{2},\ \Lambda,\ \Psi\in\mathbb{R}.$$

(b) The Logarithmic mean:

$$L = L(\Lambda, \Psi) := \frac{\Psi - \Lambda}{\ln \Psi - \ln \Lambda},\ \Lambda,\ \Psi \in \mathbb{R},\ \Lambda \neq \Psi.$$

(c) The Generalized Logarithmic-mean:

$$L_r = L_r(\Lambda, \Psi) := \left[\frac{\Psi^{r+1} - \Lambda^{r+1}}{(r+1)(\Psi - \Lambda)}\right]\ r \in \mathbb{R}\backslash\{-1, 0\},\ \Lambda, \Psi \in \mathbb{R},\ \Lambda \neq \Psi.$$

Proposition 6.1. *Let* $\Lambda, \Psi \in \mathbb{R}$ *with* $0 < \Lambda < \Psi$, *we have,*

$$\left|\frac{4(\Psi - \Lambda)}{27}\left(A^3\left(\Lambda^2, 2\Psi^2\right) + A^3(2\Lambda, \Psi)\right) + \frac{1}{4}A\left(\Lambda^4, \Psi^4\right) + \frac{2}{27}\left(A^4(2\Lambda, \Psi) + A^4(\Lambda, 2\Psi)\right) - L_5^5(\Lambda, \Psi)\right|$$
$$\leq (\Psi - \Lambda)^2\left(\frac{227}{864}\right)\left(4\Lambda^2 + \Psi^2\right).$$

Proof. The assertion follows from Theorem 4.1, $\Omega(x) = \frac{x^4}{12}$, $\alpha = s = 1$ and $\beta(0) = \beta(1) = 1$, where $|\Omega''| = x^2$ the φ-convex function $\varphi(\Lambda, \Psi) = 2\Lambda + \Psi$. □

Proposition 6.2. *Let* $\Lambda, \Psi \in \mathbb{R}$ *with* $0 < \Lambda < \Psi$, *we have,*

$$\left|\frac{10(\Psi - \Lambda)}{81}\left(A^4(\Lambda, 2\Psi) + A^4(2\Lambda, \Psi)\right) + \frac{1}{4}A\left(\Lambda^5, \Psi^5\right) + \frac{4}{81}\left(A^5(2\Lambda, \Psi) + A^5(\Lambda, 2\Psi)\right) - L_6^6(\Lambda, \Psi)\right|$$
$$\leq (\Psi - \Lambda)^2\left(\frac{1135}{2592}\right)\left(2\Lambda^3 + \Lambda^9 - \Psi^9\right).$$

Proof. The assertion follows from Theorem 4.1, $\Omega(x) = \frac{x^5}{20}$, $\alpha = s = 1$ and $\beta(0) = \beta(1) = 1$, where $|\Omega''| = x^3$ the φ-convex function $\varphi(\Lambda, \Psi) = 3\Lambda^2(\Lambda - \Psi) + 3\Lambda(\Lambda - \Psi)^2 + (\Lambda - \Psi)^3$. □

Proposition 6.3. *Let* $\Lambda, \Psi \in \mathbb{R}$ *with* $0 < \Lambda < \Psi$, *we have,*

$$\left|\frac{(\Psi - \Lambda)^2}{12} + \frac{1}{4}A\left(\Lambda^4, \Psi^4\right) + \frac{1}{6}\left(A^2(2\Lambda, \Psi) + A^2(\Lambda, 2\Psi)\right) - L_3^3(\Lambda, \Psi)\right| \leq \frac{(\Psi - \Lambda)^2}{2592}.$$

Proof. The assertion follows from Theorem 4.1, $\Omega(x) = x^2$, $\alpha = s = 1$ and $\beta(0) = \beta(1) = 1$, where $|\Omega''| = 2$ the φ-convex function $\varphi(\Lambda, \Psi) = \Lambda - \Psi$. □

Proposition 6.4. *Let* $\Lambda, \Psi \in \mathbb{R}$ *with* $0 < \Lambda < \Psi$, *we have,*

$$\left|\frac{9(\Psi - \Lambda)}{32}\left(A^{-2}(2\Lambda, \Psi) - A^{-2}(\Lambda, 2\Psi)\right) + \frac{1}{4}H^{-1}(\Lambda, \Psi) + \frac{9}{16}\left(A^{-1}(2\Lambda, \Psi) + A^{-1}(\Lambda, 2\Psi)\right) - L^{-1}(\Lambda, \Psi)\right|$$
$$\leq (\Psi - \Lambda)^2\left(\frac{227}{5184}\right)\max\left(\frac{2}{\Lambda^3}, \frac{2}{\Psi^3}\right).$$

Proof. The assertion follows from Theorem 5.2, $\Omega(x) = \frac{1}{x}$, $\alpha = 1$ and $\beta(0) = \beta(1) = 1$, where $|\Omega''| = \left|\frac{2}{x^3}\right|$ the φ-convex function $\varphi(\Lambda, \Psi) = \Lambda - \Psi$. □

Next we give two examples for definition.

Example 6.1. *Suppose* $\Omega(x) = x^3$, *then* Ω *is not convex but is* φ*-convex with* $\varphi(\Lambda, \Psi) = 3\Psi^2(\Lambda - \Psi) + 3\Psi(\Lambda - \Psi)^2 + (\Lambda - \Psi)^3$, *indeed:*

$$\begin{aligned}
\Omega(\Delta\Lambda + (1-\Delta)\Psi) &= (\Delta\Lambda + (1-\Delta)\Psi)^3 = (\Delta(\Lambda - \Psi) + \Psi)^3 \\
&= \Psi^3 + 3\Psi^2\Delta(\Lambda - \Psi) + 3\Psi\Delta^2(\Lambda - \Psi)^2 + \Delta^3(\Lambda - \Psi)^3 \\
&= \Omega(\Psi) + \Delta\left[3\Psi^2(\Lambda - \Psi) + 3\Psi\Delta(\Lambda - \Psi)^2 + \Delta^2(\Lambda - \Psi)^3\right] \\
&\leq \Omega(\Psi) + \Delta\left[3\Psi^2(\Lambda - \Psi) + 3\Psi(\Lambda - \Psi)^2 + (\Lambda - \Psi)^3\right] \\
&= \Omega(\Psi) + \Delta\varphi(\Omega(\Lambda), \Omega(\Psi)).
\end{aligned}$$

Example 6.2. *Suppose* $\Omega : [\Lambda, \Psi] \to \mathbb{R}$, $0 < \Lambda < \Psi$, *with* $\Omega(x) = 2$. *We obviously see that* Ω *is a* φ*-quasi-convex with* $\varphi(\Lambda, \Psi) = \Lambda - \Psi$.

7 Conclusion

Inequalities play a crucial role in various branches of mathematics and beyond. Several authors developed the Newton type inequalities using fractional operators and different convexities. Firstly, we have obtained new error bounds and given refinement to Newton-type inequality that will inspire many authors to proved and formulation of new inequalities. Secondly, we have established the new identity via Caputo-Fabrizio fractional integral operator. Employing this new identity of perturbed Newton-type inequalities for the s-φ-convex, and φ-quasi-convex functions. Simpson-type inequality is said to be Simpson's second formula if it consists of four points. Based on this identity, we have introduced the refinement of Simpson's second formula. Moreover, we also include the applications for some special means. In the future, scholars may expand mthis work with modified convexity and other fractional operators e.g Riemann-Liouville (RL), Caputo-Fabrizio (CF), Atangana-Baleanu (AB) and the Grunwald-Letnikov (GL)).

Acknowledgement The authors would like to acknowledge the support and contributions from all parties involved in facilitating this research. We also extend our appreciation to the institutions and individuals who provided valuable insights and assistance throughout the course of this study.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] S. I. Butt, A. O. Akdemir, M. Y. Bhatti & M. Nadeem (2020). New refinements of Chebyshev–Pólya–Szegö-type inequalities via generalized fractional integral operators. *Journal of Inequalities and Applications*, *2020*, 1–13. https://doi.org/10.1186/s13660-020-02425-6.

[2] S. I. Butt, I. Javed, P. Agarwal & J. J. Nieto (2023). Newton–Simpson-type inequalities via majorization. *Journal of Inequalities and Applications*, *2023*(1), Article ID: 16. https://doi.org/10.1186/s13660-023-02918-0.

[3] M. Caputo & M. Fabrizio (2015). A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation & Applications*, *1*(2), 73–85. https://doi.org/10.12785/pfda/010201.

[4] S. Chasreechai, M. A. Ali, S. Naowarat, T. Sitthiwirattham & K. Nonlaopon (2023). On some Simpson's and Newton's type of inequalities in multiplicative calculus with applications. *AIMS Mathematics*, *8*(2), 3885–3896. https://doi.org/10.3934/math.2023193.

[5] J. Chen & X. Huang (2017). Some new inequalities of Simpson's type for s-convex functions via fractional integrals. *Filomat*, *31*(15), 4989–4997. https://doi.org/10.2298/FIL1715989C.

[6] S. S. Dragomir, R. P. Agarwal & P. Cerone (2000). On Simpson's inequality and applications. *Journal of Inequalities and Applications*, *5*(6). https://doi.org/10.1155/S102558340000031X.

[7] T. Du, Y. Li & Z. Yang (2017). A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions. *Applied Mathematics and Computation*, *293*, 358–369. https://doi.org/10.1016/j.amc.2016.08.045.

[8] S. Erden, S. Iftikhar, P. Kumam & P. Thounthong (2024). On error estimations of Simpson's second type quadrature formula. *Mathematical Methods in the Applied Sciences*, *47*(13), 11232–11244. https://doi.org/10.1002/mma.7019.

[9] S. Gao & W. Shi (2012). On new inequalities of Newton's type for functions whose second derivatives absolute values are convex. *International Journal of Pure and Applied Mathematics*, *74*(1), 33–41.

[10] M. E. Gordji, M. R. Delavar & M. De La Sen (2016). On φ-convex functions. *Journal of Mathematical Inequalities*, *10*(1), 173–183. https://doi.org//10.7153/jmi-10-15.

[11] F. Hezenci, H. Budak & H. Kara (2021). New version of fractional Simpson type inequalities for twice differentiable functions. *Advances in Difference Equations*, *2021*, Article ID: 460. https://doi.org/10.1186/s13662-021-03615-2.

[12] S. Iftikhar, P. Kumam & S. Erden (2020). Newton's-type integral inequalities via local fractional integrals. *Fractals*, *28*(3), Article ID: 2050037. https://doi.org/10.1142/S0218348X20500371.

[13] M. Junjua, A. Qayyum, A. Munir, H. Budak, M. M. Saleem & S. S. Supadi (2024). A study of some new Hermite–Hadamard inequalities via specific convex functions with applications. *Mathematics*, *12*(3), Article ID: 478. https://doi.org/10.3390/math12030478.

[14] N. Laribi & B. Meftah (2023). 3/8-Simpson type inequalities for functions whose modulus of first derivatives and its Q-th powers are s-convex in the second sense. *Jordan Journal of Mathematics and Statistics*, *16*(1), 79–98. https://doi.org/10.47013/16.1.6.

[15] Y. L. Luke (1969). *The Special Functions and Their Approximations*. Academic Press, New York.

[16] R. L. Magin (2006). *Fractional Calculus In Bio-Engineering*. Begell House, Danbury, USA.

[17] A. M. Mercer (2003). A variant of Jensen's inequality. *Journal of Inequalities in Pure and Applied Mathematics*, *4*(4), Article ID: 73.

[18] A. Munir, A. Qayyum, H. Budak, S. Qaisar, U. Ali & S. S. Supadi (2025). A fractional version of corrected dual-Simpson's type inequality via s-convex function with applications. *Malaysian Journal of Mathematical Sciences*, *19*(1), 17–33. https://doi.org/10.47836/mjms.19.1.02.

[19] A. Munir, M. Vivas-Cortez, A. Qayyum, H. Budak, I. Faiz & S. S. Supadi (2024). Some new fractional corrected Euler-Maclaurin type inequalities for function whose second derivatives are s-convex function. *Mathematical and Computer Modelling of Dynamical Systems*, *30*(1), 543–566. https://doi.org/10.1080/13873954.2024.2356698.

[20] B. S. Nishant, S. D. Purohit, K. S. Nisar & S. R. Munjam (2024). Some fractional calculus findings associated with the product of incomplete ℵ-function and Srivastava polynomials. *International Journal of Mathematics and Computer in Engineering*, *2*, 97–116. https://doi.org/10.2478/ijmce-2024-0008.

[21] M. A. Noor, K. I. Noor & S. Iftikhar (2018). Newton inequalities for p-harmonic convex functions. *Honam Mathematical Journal*, *40*(2), 239–250. https://dx.doi.org/10.5831/HMJ.2018.40.2.239.

[22] J. Park (2013). On Simpson-like type integral inequalities for differentiable preinvex functions. *Applied Mathematical Sciences*, *7*(121), 6009–6021. http://dx.doi.org/10.12988/ams.2013.39498.

[23] S. K. Sahoo, H. Ahmad, M. Tariq, B. Kodamasingh, H. Aydi & M. De la Sen (2021). Hermite–Hadamard type inequalities involving k-fractional operator for $(\bar{h}, m)$-convex functions. *Symmetry*, *13*(9), Article ID: 1686. https://doi.org/10.3390/sym13091686.

[24] W. Saleh, A. Lakhdari, T. Abdeljawad & B. Meftah (2023). On fractional biparameterized Newton-type inequalities. *Journal of Inequalities and Applications*, *2023*(1), Article ID: 122. https://doi.org/10.1186/s13660-023-03033-w.

[25] Z. Şanlı (2022). Simpson type conformable fractional inequalities. *Journal of Function Spaces*, *2022*(1), Article ID: 5707887. https://doi.org/10.1155/2022/5707887.

[26] M. Z. Sarikaya & H. Yaldiz (2015). On Hermite-Hadamard type inequalities for ϕ-convex functions via fractional integrals. *Malaysian Journal of Mathematical Sciences*, *9*(2), 243–258.

[27] R. Singh, J. Mishra & V. K. Gupta (2023). Dynamical analysis of a tumor growth model under the effect of fractal fractional Caputo-Fabrizio derivative. *International Journal of Mathematics and Computer in Engineering*, *1*(1), 115–126. https://doi.org/10.2478/ijmce-2023-0009.

[28] P. Siricharuanun, S. Erden, M. A. Ali, H. Budak, S. Chasreechai & T. Sitthiwirattham (2021). Some new Simpson's and Newton's formulas type inequalities for convex functions in quantum calculus. *Mathematics*, *9*(16), Article ID: 1992. https://doi.org/10.3390/math9161992.

[29] J. Soontharanon, M. A. Ali, H. Budak, K. Nonlaopon & Z. Abdullah (2022). Simpson's and Newton's type inequalities for (α, m)-convex functions via quantum calculus. *Symmetry*, *14*(4), Article ID: 736. https://doi.org/10.3390/sym14040736.

[30] M. Vivas Cortez & Y. Rangel Oliveros (2020). An inequality related to s-φ-convex functions. *Applied Mathematics & Information Sciences*, *14*(1), 151–154. https://doi.org/10.18576/amis/140119.

[31] X. You, F. Hezenci, H. Budak & H. Kara (2022). New Simpson type inequalities for twice differentiable functions via generalized fractional integrals. *AIMS Mathematics*, *7*(3), 3959–3971. https://doi.org/10.3934/math.2022218.